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EIGENVALUES OF THE ADIN-ROICHMAN MATRICES

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Abstract. We find the spectrum of the Walsh-Hadamard type matrices defined by R.Adin and Y.Roichman in their recent work on character formulas and descent sets for the symmetric group.

1. Introduction

Adin and Roichman described in [1] a general framework for various character formulas for representations of the symmetric group. A key ingredient in their description is a family of matrices- $(A_n)_{n\geq 0}$ and $(B_n)_{n\geq 0}$which are defined, recursively, by $A_0 = B_0 = (1)$ and for $n \geq 1$,

$$A_n = \begin{pmatrix} A_{n-1} & A_{n-1} \\ A_{n-1} & -B_{n-1} \end{pmatrix}, \quad B_n = \begin{pmatrix} A_{n-1} & A_{n-1} \\ 0 & -B_{n-1} \end{pmatrix}.$$

The matrix A_n was shown to connect between combinatorial objects of various types and character values. It has been shown in [1] that A_n is invertible, and hence the character formulas may be inverted, yealding formulas for counting combinatorial objects with a given descent set using character values.

Adin and Roichman asked the more subtle question of finding the eigenvalues of A_n and B_n. They made the following conjecture:

Conjecture 1. [1, Conjecture 4.10]

(i) *The roots of the characteristic polynomial of A_n are in $2:1$ correspondence with the compositions of n: each composition $\mu = (\mu_1, \ldots, \mu_t)$ of n corresponds to a pair of eigenvalues $\pm\sqrt{\pi_\mu}$ of A_n, where*

$$\pi_\mu := \prod_{i=1}^{t} (\mu_i + 1).$$

(ii) *Similarly, the roots of the characteristic polynomial of B_n are in $2:1$ correspondence with the compositions of n: each composition $\mu = (\mu_1, \ldots, \mu_t)$ of n corresponds to a pair of eigenvalues $\pm\sqrt{\pi'_\mu}$ of B_n, where*

$$\pi'_\mu := \prod_{i=1}^{t-1} (\mu_i + 1).$$

We will prove this conjecture (see Theorem 20 below). Our method of proof is as follows: We conjugate the matrices A_n and B_n by a combinatorially defined matrix U_n, and get a lower anti-triangular matrix, i.e. a matrix with zeros above the secondary diagonal. The area below the secondary diagonal in the conjugated matrices is quite sparse, and we show that a permutation can be chosen, so that

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after conjugating with the corresponding permutation matrix we get matrices which are lower-triangular in blocks of size 2×2. From this form, the eigenvalues can be easily obtained.

The rest of this paper is organized as follows: in section 2 we give some definitions, and recall the non-recursive definition of A_n and B_n from [1]. In section 3, we outline in more detail the strategy of the proof. In section 4, we describe the conjugation of A_n and B_n into lower anti-triangular matrices. In section 5, we find the suitable permutation, and in section 6 we use it to prove the conjecture. In the final section we give an equivalent, non-recursive description of the permutation involved in the proof, and also describe it in terms of the Thue-Morse sequence.

2. Preliminaries

Let us recall some of the definitions in [1]. We use the notation $[n] = \{1, 2, .., n\}$ and $[a, b] = \{i \in \mathbb{Z} | a \le i \le b\}$. A nonempty set of the form $[a, b]$ is called an *interval*. Given intervals I_1 and I_2, I_1 is called a *prefix* of I_2 if $\min I_1 = \min I_2$ and $I_1 \subseteq I_2$.

Given a set $I \subseteq [n]$, the *runs of* I are the maximal intervals contained in I. They are denoted, in ascending order, by $I_1, I_2, ...$. For example, if $I = \{2, 4, 5\}$ then $I_1 = \{2\}$ and $I_2 = \{4, 5\}$.

Definition 2. Given sets $I, J \subseteq [n]$, let us write $I \gg J$ if each run of $I \cap J$ is a prefix of a run of I. (Note that this is not an order relation).

Let us now describe the non-recursive definition of A_n and B_n. It is convenient to index the rows and columns of the $2^n \times 2^n$ matrices A_n and B_n by subsets of $[n]$. We order the subsets of $[n]$ linearly by the lexicographical order, as described in [1]. An equivalent definition of the lexicographical order comes from the following function:

Definition 3. Let $r_n : P([n]) \to [2^n]$ be given by the binary representation,

$$r_n(A) = 1 + \sum_{i \in A} 2^{i-1}.$$

Note that $A \le B$ with respect to the lexicographical order if and only if $r_n(A) \le r_n(B)$.

Lemma 4. *The matrices A_n and B_n are given by*

$$A_n(I, J) = \begin{cases} (-1)^{|I \cap J|} & \text{if } I \gg J \\ 0 & \text{otherwise} \end{cases}$$

and

$$B_n(I, J) = \begin{cases} (-1)^{|I \cap J|} & \text{if } I \gg J \text{ and } n \notin I \setminus J \\ 0 & \text{otherwise} \end{cases}$$

The lemma is proved in [1, lemma 4.8].

3. Strategy of the proof

Definition 5. Let U_n be the matrix $U_n(I, J) = \begin{cases} 1 & I \supseteq J \\ 0 & \text{otherwise} \end{cases}$

Note that U_n is the transpose of the matrix Z_n defined in [1].

Definition 6. A matrix A of size $m \times m$ is called *lower anti-triangular* if $A_{i,j} = 0$ for all i, j satisfying $i \leq n - j$.

We will show that the matrices $U_n A_n U_n^{-1}$ and $U_n B_n U_n^{-1}$ are (when rows and columns are written in lexicographical order) anti-triangular. For example,

$$U_3 A_3 U_3^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 4 & 0 & 2 & 0 & 2 & 0 & 0 & 0 \end{pmatrix}$$

Furthermore, we will see that $U_n A_n U_n^{-1}$ can be conjugated by a permutation matrix, such that the resulting matrix is block-triangular with blocks of size 2×2.

For example, for $n = 3$ we may take the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 3 & 2 & 7 & 4 & 5 & 8 & 1 \end{pmatrix}.$$

If P is the corresponding permutation matrix, then

$$PU_3 A_3 U_3^{-1} P^{-1} = \begin{pmatrix} 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

It is easy to deduce the eigenvalues of A_n from this form.

4. Conjugation by U_n

Definition 7. Let $A'_n = U_n A_n (U_n)^{-1}$ and $B'_n = U_n B_n (U_n)^{-1}$.

We will prove below that, A'_n and B'_n are anti-triangular. Some other properties of these matrices may be observed. For example, for $n = 9$ and $I = \{1, 2, 3, 6, 7, 9\}$, we may note that $A'_9(I, J) \neq 0$ only for

$$J = \{4,5,8\}, \{2,4,5,8\}, \{3,4,5,8\}, \{4,5,7,8\}, \{2,4,5,7,8\}, \{3,4,5,7,8\}$$

that is, only for sets of the form $\bar{I} \cup E$ where $E \subseteq I$ and E does not contain a minimal element of a run of I, nor does it contain two consecutive elements.

Definition 8. For a set $I \subseteq [n]$, let us denote $\pi(I) = \prod_i (n_i + 1)$, where n_i is the size of the ith run I_i.

Example 9. $\pi(\{1, 2, 3, 5\}) = (3 + 1)(1 + 1) = 8$.

Lemma 10. *We have*

(1) $A'_n(I, J) = 0$ *unless* $\bar{I} \subseteq J$ *(where* $\bar{I} = [n] \setminus I$*).*
(2) $A'_n(I, \bar{I}) = \pi(I)$

(3) $A'_n(I, \bar{I} \cup E) = 0$ *if* $E \subseteq I$ *and* E *contains a minimal element of an interval of* I.
(4) $A'_n(I, \bar{I} \cup E) = 0$ *if* $E \subseteq I$ *and* $i, i+1 \in E$ *for some* i.

Proof. By Möbius inversion (see [3] and [1, section 5]), the inverse of U_n is given by

$$(U_n)^{-1}(I, J) = \begin{cases} (-1)^{|I \setminus J|} & \text{if } I \supseteq J \\ 0 & \text{otherwise} \end{cases}$$

Hence we have, by definition of matrix multiplication,

$$\begin{aligned} A'_n(I, L) &= \sum_{J,K \subseteq [n]} U_n(I, J) \cdot A'_n(J, K) \cdot (U_n^{-1})(K, L) \\ &= \sum_{J,K: I \supseteq J, J \gg K, K \supseteq L} (-1)^{|J \cap K| + |K \setminus L|} \end{aligned}$$

We will use the last formula in the proof of each claim. Given I, J, K and some $x \in [n]$, let us say that *we may toggle* x *in* K if for each $K \subseteq [n] \setminus \{x\}$, the contributions of (J, K) and $(J, K \cup \{x\})$ to the above some cancel out. Similarly, given I, K and L, we will say that *we may toggle* x *in* J if for each $J \subseteq [n] \setminus \{x\}$, the contributions of (J, K) and $(J \cup \{x\}, K)$ to the above sum cancel out.

(1) Let us assume that $\bar{I} \nsubseteq L$, and let $x \in [n]$ be such that $x \notin I$ and $x \notin L$. For each J in the sum $\sum_{J,K: I \supseteq J, J \gg K, K \supseteq L} (-1)^{|J \cap K| + |K \setminus L|}$, we have $x \notin J$. We may toggle x in K, since for $K \subseteq [n] \setminus \{x\}$, K and $K \cup \{x\}$ have the same intersection with J. Hence, the entire sum cancels out, i.e. $A'_n(I, L) = 0$.
(2) We have

$$A'_n(I, I) = \sum_{J,K: I \supseteq J, J \gg K, K \supseteq \bar{I}} (-1)^{|J \cap K| + |K \cap I|}.$$

For each $J \subsetneq I$, the sum over J is 0, since we may take $x \in I \setminus J$ and toggle x in K (as in the previous case, adding x to K does not change the intersection $K \cap J$). Hence only $I = J$ contributes to the sum and we get

$$A'_n(I, I) = \sum_{K: I \gg K, K \supseteq \bar{I}} (-1)^{|J \cap K| + |K \cap I|} = \pi(I)$$

(3) Suppose that $x \in E$ is a minimal element of an interval in I. For each J participating in the sum

$$A'_n(I, \bar{I} \cup E) = \sum_{J,K: I \supseteq J, J \gg K, K \supseteq \bar{I} \cup E} (-1)^{|J \cap K| + |K \cap I \cap \bar{E}|}$$

we have $x - 1 \notin I \Rightarrow x - 1 \notin J$, whereas $x \in K$ (because $x \in E$). Hence, we may toggle x in J (adding x to J may only extend one interval in J one place to the left, and removing x may only shrink one interval by one place from the left, hence $J \gg K \Leftrightarrow J \cup \{x\} \gg K$), and the whole sum is 0.
(4) If $i, i+1 \in E$ then in the sum for $A'_n(I, \bar{I} \cup E)$, for each K in the sum we have $i, i+1 \in I \cap K$ and we may toggle $i+1$ in J.

□

Definition 11. Let $I, J \subseteq [n]$.

(1) We write $J \preceq I$ if $J \subseteq I$ and J does not contain any minimal element of a run of I, nor does it contain two consecutive elements.
(2) We write $I \curvearrowright J$ if $J = \bar{I} \cup E$ for some $E \preceq I$.

Corollary 12.
(1) *A'_n is lower anti-triangular.*
(2) *$A'_n(I,J) \neq 0$ only if $I \curvearrowright J$.*

This follows immediately from lemma 10.

Similar results hold for B'_n:

Definition 13. For a set $A \subseteq [n]$, let us denote $\pi'_n(A) = \prod_i (n_i + 1)$, where n_i is the size of the ith run I_i, and the product excludes the run containing n, if it exists.

Example 14. $\pi'_8(\{1,2,4,5,7,8\}) = (2+1)(2+1) = 9$ and $\pi'_8(\{1,2,4,6,7) = (2+1)(1+1)(2+1) = 18$.

Lemma 15. *We have*
(1) *$B'_n(I,J) = 0$ unless $\bar{I} \subseteq J$. (where $\bar{I} = [n] \setminus I$).*
(2) *$B'_n(I,\bar{I}) = \pi'_n(I)$*
(3) *$B'_n(I,\bar{I} \cup E) = 0$ if $E \subseteq I$ and E contains a minimal element of a run of I.*
(4) *$B'_n(I,\bar{I} \cup E) = 0$ if $E \subseteq I$ and $i, i+1 \in E$ for some i.*

Proof. The proof goes along the lines of the proof of lemma 10. We have

$$B'_n(I,L) = \sum_{J,K: I \supseteq J, J \gg K, K \supseteq L, n \notin J \setminus K} (-1)^{|J \cap K| + |K \setminus L|}$$

We repeat the arguments in the above proof:

(1) We assume that $\bar{I} \not\subseteq L$, and let $x \in [n]$ be such that $x \notin I$ and $x \notin L$. Since for each J in the sum, $x \notin J$, adding or removing x from K does not change $J \setminus K$. Hence, we may still toggle x in K, and get $B'_n(I,L) = 0$.
(2) We have

$$B'_n(I,I) = \sum_{J,K: I \supseteq J, J \gg K, K \supseteq \bar{I}, n \notin J \setminus K} (-1)^{|J \cap K| + |K \cap I|}.$$

If $J \subsetneq I$, we may still take $x \in I \setminus J$ and toggle x in K (adding or removing x from K will not change $J \setminus K$). Hence we may take $I = J$ in the sum:

$$B'_n(I,I) = \sum_{K: I \gg K, K \supseteq \bar{I}, n \notin I \setminus K} (-1)^{|J \cap K| + |K \cap I|} = \pi'_n(I).$$

(3) We have

$$B'_n(I,\bar{I} \cup E) = \sum_{J,K: I \supseteq J, J \gg K, K \supseteq \bar{I} \cup E, n \notin J \setminus K} (-1)^{|J \cap K| + |K \cap I \cap \bar{E}|}.$$

If $x \in E$ is a minimal element of a run of I, then for each K participating in the sum for $B_n(I,\bar{I} \cup E)$ we have $x \in K$, and toggling x in J does not change $J \setminus K$.
(4) Again, if $i, i+1 \in E$ then $i+1 \in K$ and we may still toggle $i+1$ in J.

□

Corollary 16.
(1) *B'_n is lower anti-triangular.*
(2) *$B'_n(I,J) \neq 0$ only if $I \curvearrowright J$.*

5. Conjugation by a permutation matrix

We have shown that A_n is conjugate to a matrix A'_n which satisfies some nice properties: It is anti-triangular, its anti-diagonal elements are given by $A_n(I,\bar{I}) = \pi(I)$, and it is sparse: $A'_n(I,J) \neq 0$ only if $I \curvearrowright J$. We will use all these properties to find a suitable permutation matrix for further conjugating A'_n into a 2×2 block-triangular matrix. The same permutation will also conjugate B'_n into a block matrix of the same type.

Lemma 17. *There exists a one-to-one function $\sigma_n : [2^n] \to P([n])$ such that:*

(1) *For all $1 \le i \le 2^{n-1}$, $\sigma_n(2i) = \overline{\sigma_n(2i-1)}$*
(2) *For all $1 \le i \le 2^{n-1}, 1 \in \sigma_n(2i-1)$*
(3) *If $\sigma_n(i) \curvearrowright \sigma(j)$ then $\sigma_n(i) = \overline{\sigma_n(j)}$ or $j \le i$.*

Simply put, the lemma states that we can list the subsets of $[n]$ in pairs of complementing sets, such that when a set I is listed, all the sets J such that $I \curvearrowright J$, except possibly $\overline{I}$, have already been listed.

Example 18. For $n = 3$, we may take the following function:

i	$\sigma_3(2i-1)$	$\sigma_3(2i)$
1	$\{1,3\}$	$\{2\}$
2	$\{1\}$	$\{2,3\}$
3	$\{1,2\}$	$\{3\}$
4	$\{1,2,3\}$	$\emptyset$

In fact, this is the only possible function, but for larger values of n it is not always unique. Note, however that for any n we must have $\sigma_n(1) = \{1,3,5,...\}$.

Proof. We shall construct a function σ_n satifying the above conditions explicitely.

The construction is recursive: For $n = 1$ we define $\sigma_1(1) = \{1\}, \sigma_1(2) = \emptyset$.

Let us assume that $\sigma_1, ..., \sigma_{n-1}$ have been defined.

First we define the value of $\sigma_n(j)$ for $1 \le j \le 2^{n-1}$ by:

$$\begin{aligned}
\sigma_n(2i-1) &= \{1\} \cup (\sigma_{n-1}(2i) + 1) \\
\sigma_n(2i) &= \sigma_{n-1}(2i-1) + 1 \\
(1 \le i \le 2^{n-2}) &
\end{aligned}$$

Note that all the pairs of sets in this half-list have 1 in one set and 2 in the other.

We define the next 2^{n-2} values by

$$\begin{aligned}
\sigma_n(2^{n-1} + 2i - 1) &= \{1,2\} \cup (\sigma_{n-2}(2i) + 2) \\
\sigma_n(2^{n-1} + 2i) &= \sigma_{n-2}(2i-1) + 2 \\
(1 \le i \le 2^{n-3}) &
\end{aligned}$$

and in general,

$$\begin{aligned}
\sigma_n(2^n - 2^{n-k} + 2i - 1) &= [k+1] \cup (\sigma_{n-k-1}(2i) + k + 1) \\
\sigma_n(2^n - 2^{n-k} + 2i) &= \sigma_{n-k-1}(2i-1) + k + 1
\end{aligned}$$

for all $0 \le k \le n-2, 1 \le i \le 2^{n-k-2}$.

Finally, we define

$$\begin{aligned}\sigma_n(2^n-1) &= [n] \\ \sigma_n(2^n) &= \emptyset.\end{aligned}$$

Let us call the sets defined at the k-th stage (i.e. the sets at places $2^n - 2^{n-k} + 1, .., 2^n - 2^{n-k} + 2^{n-k-1}$) *the sets of the k-th chunk*. Since all the functions σ_i are one-to-one, the k-th chunk consists of sets that contain $[k+1]$ but don't contain $\{k+2\}$, and the complements of these sets (which are exactly the sets whose minimum is $k+2$).

Let us prove that the conditions are satisfied: the first two, $\sigma_n(2i) = \overline{\sigma_n(2i-1)}$ and $1 \in \sigma_n(2i-1)$, are easy to check. For the third one, we look again at

$$I := \sigma_n(2^n - 2^{n-k} + 2i - 1) = [k+1] \cup (\sigma_{n-k-1}(2i) + k + 1)$$

If $I \curvearrowright J$ and $J \neq \bar{I}$, then we may write $J = \bar{I} \cup E$ for some $\emptyset \neq E \preceq I$. If $[k+1] \cap E = \emptyset$, then $\min J = k+2$, hence J also belongs to the kth chunk, and by the induction hypothesis, since $(I \setminus [k+1]) - (k+1) \curvearrowright J - (k+1)$, J is equal to $\sigma_n(2^n - 2^{n-k} + 2j)$ for some $1 \leq j < i$. If $[k+1] \cap E \neq \emptyset$, then let $l = \min E$. We have $1 < l \leq k+1$ (1 cannot be an element of E since $E \preceq I$) and $\bar{I} \cup E$ belongs to the $(l-2)$nd chunk, hence (since $l - 2 < k$) appears before I in the list.

Next, we consider

$$J := \sigma_n(2^n - 2^{n-k} + 2i) = \sigma_{n-k-1}(2i-1) + k + 1$$

Note that $\min J = k+2$. Given $\emptyset \neq E \preceq J$, $\bar{J} \cup E$ contains $[k+1]$ and does not contain $k+2$. We have $k+2 \notin E$, hence $1 \notin E - (k+1)$. Also, $k+2 \in J$ and by the induction hypothesis $K := ([n-k-1] \setminus (J - (k+1))) \cup (E - (k+1))$ appears before $J - (k+1)$ in the list σ_{n-k-1}. Hence, $\bar{J} \cup E = [k+1] \cup (K + k + 1)$ appears before $J = (J - (k+1)) + k + 1$ in σ_n. □

6. Eigenvalues of A_n and B_n

Let us take σ_n as in lemma 17 and view σ_n as a permutation on $[2^n]$ (using, as usual, the lexicographical order on $P([n])$). Let P_n be the permutation matrix corresponding to σ_n, i.e. $P_n(i,j) = \delta_{\sigma_n(i),j}$, and let $A''_n = P_n A'_n P_n^{-1}$

We have $A''_n(i,j) = P_n A'_n P_n^{-1}(i,j) = A'_n(\sigma_n(i), \sigma_n(j))$, hence (by corollaries 12 and 16, and lemma 17) $A''_n(i,j) = 0$ if $j > i$ and $\{i,j\}$ is not of the form $\{2t+1, 2t+2\}$. Hence, A''_n is lower triangular in 2×2 blocks.

The blocks on the main diagonal of A''_n are in correspondence with subsets of $I \subseteq [n]$ satisfying $1 \in I$. To such a set corresponds the block

$$\begin{pmatrix} 0 & \pi(I) \\ \pi(\bar{I}) & 0 \end{pmatrix}$$

The characteristic polynomial of this block is $t^2 - \pi(I)\pi(\bar{I})$. Hence the characteristic polynomial of A_n is

$$\det(tI - A_n) = \det(tI - A''_n) = \prod_{1 \in I \subseteq [n]} \left(t^2 - \pi(I)\pi(\bar{I})\right)$$

Similarly, let us define $B''_n = P_n B'_n P_n^{-1}$, and again B''_n is lower triangular in 2×2 blocks, the blocks on the main diagonal are in one-to-one correspondence with subsets $1 \in I \subseteq [n]$, and the block corresponding to such I is

$$\begin{pmatrix} 0 & \pi'(I) \\ \pi'(\bar{I}) & 0 \end{pmatrix}$$

Thus,

$$\det(tI - B_n) = \det(tI - B''_n) = \prod_{1 \in I \subseteq [n]} \left(t^2 - \pi'(I)\pi'(\bar{I})\right)$$

Let us note that there is a one-to-one correspondence between sets $1 \in I \subseteq [n]$ and compositions of n:

Definition 19. Given a set I satisfying $1 \in I \subseteq [n]$, let $n_1, n_2, ...$ be the sizes of the runs $I_1, I_2, ..$ of I, and let $m_1, m_2, ...$ be the sizes of the runs $\bar{I}_1, \bar{I}_2, ...$ of $\bar{I}$. Let $\mu_n(I)$ be the composition $(n_1, m_1, n_2, m_2, ...)$ of n.

For example, $\mu_8(\{2, 4, 5\}) = (1, 1, 1, 2, 3)$.

The correspondence μ_n satisfies $\pi(\mu_n(I)) = \pi(I)\pi(\bar{I})$ and $\pi'(\mu_n(I)) = \pi'_n(I)\pi'_n(\bar{I})$. We conclude:

Theorem 20. *We have*

- $\det(tI - A_n) = \prod_\mu \left(t^2 - \pi(\mu)\right)$
- $\det\left(tI - B_n\right) = \prod_\mu \left(t^2 - \pi'(\mu)\right)$

The products extend over all compositions μ of n.

This proves conjecture 1.

7. A closer look at σ_n

The function $\sigma_n : [2^n] \to P([n])$ has been defined recursively in the proof of lemma 17. We will now give a non-recursive definition. For that matter, it is more convenient to look at the permutation $\sigma_n r_n : P([n]) \to P([n])$ (recall the definition of r_n in section 2).

Theorem 21. *The permutation $\sigma_n r_n$ is given by*

$$t \in \sigma_n r_n(I) \Leftrightarrow |\left(\{1\} \cup [n - t + 2, n]\right) \setminus I| \equiv 1 \mod 2$$

Proof. Let us prove by induction on n. For $n = 1$, we have $\sigma_1 r_1(\emptyset) = \sigma_1(1) = \{1\}$ and $\sigma_1 r_1(\{1\}) = \sigma_1(2) = \emptyset$, and accordingly, $|\{1\} \setminus \emptyset| = 1$ and $|\{1\} \setminus \{1\}| = 0$.

Suppose that the claim is true for $1, .., n - 1$. Recall the recursive definition

$$\sigma_n(2^n - 2^{n-k} + 2i - 1) = [k + 1] \cup (\sigma_{n-k-1}(2i) + k + 1)$$

Let $2i - 1 = r_{n-k}(J)$, for some $J \subseteq [n - k - 1]$ (note that $1 \notin J$).
Since $2^n - 2^{n-k} = 2^{n-k} + 2^{n-k+1} + .. + 2^{n-1}$, we have

$$2^n - 2^{n-k} + 2i - 1 = r_n(J \cup [n - k + 1, n]).$$

Let

$$I := J \cup [n - k + 1, n].$$

Note that $n - k \notin J$ and $n - k \notin I$.
We have

$$\begin{aligned} \sigma_n r_n(I) &= [k + 1] \cup (\sigma_{n-k-1}(2i) + k + 1) \\ &= [k + 1] \cup (\sigma_{n-k-1} r_{n-k-1}(\{1\} \cup J) + k + 1) \end{aligned}$$

For all $t \in [n]$,

- If $t \le k+1$ then (since $1 \notin I$),
$$|(\{1\} \cup [n-t+2,n]) \setminus I| = 1$$
and $t \in \sigma_n r_n(I)$.
- If $t = k+2$, then (since $n-k \notin I$),
$$|(\{1\} \cup [n-t+2,n]) \setminus I| = 2$$
and (since $1 \notin \sigma_{n-k-1}(2i)$), $t \notin \sigma_n r_n(I)$.
- If $t > k+2$ then
$$|(\{1\} \cup [n-t+2,n]) \setminus I| = |(\{1\} \cup [n-t+2,n-k]) \setminus J)|.$$
by the induction hypothesis,
$$\begin{aligned} t \in \sigma_n r_n(I) && \Leftrightarrow \\ t-(k+1) \in \sigma_{n-k-1} r_{n-k-1}(\{1\} \cup J) && \Leftrightarrow \\ |(\{1\} \cup [(n-k-1-(t-(k+1))+2, n-k-1]) \setminus (\{1\} \cup J)| \equiv 1 \mod 2 && \Leftrightarrow \\ |[n-t+2,n-k-1] \setminus J| \equiv 1 \mod 2 && \Leftrightarrow \\ |(\{1\} \cup [n-t+2,n]) \setminus I| \equiv 1 \mod 2 && \end{aligned}$$
as desired (in the last stage we used the fact that $n-k \notin I$ and $[n-k+1,n] \subseteq I$).

Also, $\sigma_n r_n([2,n]) = \sigma_n(2^n - 1) = [n]$, and $|(\{1\} \cup [n-t+2,n]) \setminus [2,n]| \equiv 1 \mod 2$ for all t, as desired. Thus, we have verified the claim for all $I \subseteq [n]$ such that $1 \in I$. The remaining cases follow easily from the property $\sigma_n(2i) = \overline{\sigma_n(2i-1)}$. $\square$

Another description of σ_n has to do with the Thue-Morse sequence. Let us recall the definition of the sequence: It is a binary sequence $(t_n)_{n\ge 0}$, obtained as the limit of the finite words w_n, where $w_0 = 0$ and $w_{n+1} = w_n\overline{w_n}$ for $n \ge 0$. For example, $w_1 = 01, w_2 = 0110, w_3 = 01101001$, hence the first terms of the sequence are $0, 1, 1, 0, 1, 0, 0, 1, ..$.

The term t_n of the Thue-Morse sequence is equal, modulo 2, to the sum of the digits in the binary expansion of n (See [2]).

Let us encode the function σ_n with n binary words of length 2^n:

Definition 22. For $1 \le i \le n$, let $W_{n,i}$ be the binary word $w^n_{i,1} w^n_{i,2} ... w^n_{i,2^n}$ where $w^n_{i,j} = \begin{cases} 1 & i \in \sigma_n(j) \\ 0 & \text{otherwise} \end{cases}$.

Then $W_{n,i}$ is given in terms of the Thue-Morse sequence:

Corollary 23. $W_{n,i} = \begin{cases} (t_0\overline{t_0})^{2^{n-i}} (t_1\overline{t_1})^{2^{n-i}} ... (t_{2^{i-1}-1}\overline{t_{2^{i-1}-1}})^{2^{n-i}} & 2|i \\ (\overline{t_0}t_0)^{2^{n-i}} (\overline{t_1}t_1)^{2^{n-i}} ... (\overline{t_{2^{i-1}-1}}t_{2^{i-1}-1})^{2^{n-i}} & 2 \nmid i \end{cases}$

Proof. Since $w^n_{i,2j-1} = \overline{w^n_{i,2j}}$, it is enough to check the equality only on the letters of $W_{n,i}$ with odd index.

Suppose that $2|i$. By theorem 21, for $I \subseteq [n]$ such that $1 \notin I$,
$$\begin{aligned} i \in \sigma_n(r_n(I))) && \Leftrightarrow \\ |[n-i+2,n] \setminus I| \equiv 0 \mod 2 && \Leftrightarrow \\ |[n-i+2,n] \cap I| \equiv 1 \mod 2 && \Leftrightarrow \\ t_{\lfloor \frac{r_n(I)-1}{2^{n+1-i}} \rfloor} = 1 && \end{aligned}$$

Similarly, if $2 \nmid i$, we get that

$$i \in \sigma_n(r_n(I))) \Leftrightarrow t_{\lfloor \frac{r_n(I)-1}{2^{n+1-i}} \rfloor} = 0$$

Since $r_n(I)$ may assume any odd value between 1 and $2^n - 1$, the proof is complete. $\square$

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